JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 45, No. 8, August 2022

Engineering Notes

Method for Pointing Spacecraft While Minimizing Change in Orientation

Patrick McKeen* and Kerri Cahoy†
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

https://doi.org/10.2514/1.G006429

Downloaded by 2601:184:4980:fb90:dd06:a50f:2d31:1e1a on May 21, 2026 | http://arc.aiaa.org | DOI: 10.2514/1.G006429

I. Introduction

ATTITUDE determination and control systems are vital to many spacecraft. Spacecraft may need to point imagers at targets or solar panels at the sun. In this Note, we devise a set of formulas that consider all orientations that could meet such single-axis or reduced attitude pointing goals, including those that minimize slew or avoid a "keep-out" zone.

Many spacecraft with active attitude control attempt to maintain a specific three-dimensional (3D) orientation in space [1–5]. However, in many cases, controlling a spacecraft to a specific 3D orientation is unnecessary. Consider a satellite with a nadir-pointing camera. If it points nadir, the goal is met, independent of rotation around the nadir vector. This consideration, called "reduced attitude," has been used in a number of approaches. In [6], this reduction is used to derive a variety of proportional–derivative (PD) controllers. Various formulas for the kinematics, dynamics, and control of reduced attitude systems are shown in [7]. A technique is proposed in [8] for an underactuated system using reduced attitude to point and avoid a keep-out zone. A reduced attitude control law robust to various uncertainties is shown in [9]. In [10], the reduced attitude system is extended to higher dimensions to derive a control law that stabilizes the full attitude simultaneously. A trajectory planner is developed in [11] that uses reduced attitude and meets complex dynamic constraints. In this Note, we derive and explore formulas that describe the set of all orientations that meet reduced attitude goals. We also consider a simple distance metric [Eq. (10)] based on the minimum angle between orientations. The distance metric is used to derive a simple formula for the orientation meeting the reduced attitude goal that has minimum slew from an arbitrary orientation. Additionally, the use of the reduced attitude set to find acceptable single-axis slews that avoid pointing toward a keep-out zone is identified. These formulas and the relevant pointing methods are outlined in an accompanying GitHub repository: https://github.com/patrickmckeen/quaternion-pointing. Finally, the formulas derived are applied to two application cases: the minimum slew from an arbitrary orientation to point an antenna at a ground station, and identifying single-axis slews to a pointing goal that avoids a keep-out zone.

Received 4 September 2021; revision received 17 March 2022; accepted for publication 28 March 2022; published online Open Access 24 May 2022. Copyright © 2022 by Patrick McKeen. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. All requests for copying and permission to reprint should be submitted to CCC at www.copyright.com; employ the eISSN 1533-3884 to initiate your request. See also AIAA Rights and Permissions www.aiaa.org/randp.

*Ph.D. Candidate, Department of Aeronautics and Astronautics, 77 Massachusetts Avenue; pmckeen@mit.edu (Corresponding Author).

†Associate Professor, Department of Aeronautics and Astronautics, 77 Massachusetts Avenue. Associate Fellow AIAA.

II. Derivation

There are many different notations and conventions for quaternions. This work will use the notation from [3]. Quaternions (which are all unit quaternions) will be bolded, such as $\boldsymbol{q} = [\boldsymbol{q}_{1:3}^T \quad q_4]^T$, while vectors (except those representing the vector part of a quaternion like $\boldsymbol{q}_{1:3}$) will be bolded and have a diacritic, such as $\boldsymbol{x}$ or $\hat{\boldsymbol{x}}$. The symbol "$*$" will indicate the conjugate of a quaternion. A unit quaternion representing a rotation of angle ψ around axis $\hat{\boldsymbol{e}}$ is [3,12]

$$\boldsymbol{q}(\hat{\boldsymbol{e}}, \psi) = \begin{bmatrix} \hat{\boldsymbol{e}}^T \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}^T \tag{1}$$

Consider a body-fixed unit vector $\hat{\boldsymbol{v}}$, such as a camera boresight. The goal is to orient the body so that $\hat{\boldsymbol{v}}$ aligns with a unit vector in the world frame $\hat{\boldsymbol{u}}$ (such as the direction toward an imaging target). A value that will be useful in this derivation is $\theta = \arccos(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}})$, which corresponds to the angle between $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{u}}$ if the coordinate axes of the body and world frames were aligned. It should be noted that $0 \le \theta \le \pi$,[‡] $\|\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}}\| = \sin\theta$, and $\|\hat{\boldsymbol{v}} + \hat{\boldsymbol{u}}\| = \sqrt{2(1+\cos\theta)} = 2\cos(\theta/2)$.

We seek the set of unit quaternions that rotate $\hat{\boldsymbol{v}}$ onto $\hat{\boldsymbol{u}}$. Specifically, we want $\boldsymbol{q}$ such that $\hat{\boldsymbol{u}} = A(\boldsymbol{q}^*)\hat{\boldsymbol{v}}$, where [3,13]

$$A(\boldsymbol{q}^*)\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}(q_4^2 - \|\boldsymbol{q}_{1:3}\|^2) + 2\boldsymbol{q}_{1:3}(\hat{\boldsymbol{v}} \cdot \boldsymbol{q}_{1:3}) + 2q_4\boldsymbol{q}_{1:3} \times \hat{\boldsymbol{v}} \tag{2}$$

We see from Eq. (2) that $\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}} = 2(\hat{\boldsymbol{v}} \cdot \boldsymbol{q}_{1:3})(\hat{\boldsymbol{v}} \times \boldsymbol{q}_{1:3}) + 2q_4(\boldsymbol{q}_{1:3} - \hat{\boldsymbol{v}}(\hat{\boldsymbol{v}} \cdot \boldsymbol{q}_{1:3}))$. and $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}} = \cos\theta = q_4^2 - \|\boldsymbol{q}_{1:3}\|^2 + 2(\hat{\boldsymbol{v}} \cdot \boldsymbol{q}_{1:3})^2$. Solving for $\hat{\boldsymbol{v}} \times \boldsymbol{q}_{1:3}$ and $\hat{\boldsymbol{v}} \cdot \boldsymbol{q}_{1:3}$ and substituting into Eq. (2) (knowing $\|\boldsymbol{q}\|^2 = 1$) yields

$$\begin{aligned} \hat{\boldsymbol{u}} &= -\hat{\boldsymbol{v}} \pm \frac{-q_4\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}} + (1 + \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}})\boldsymbol{q}_{1:3}}{\sqrt{(1/2)(1 + \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}} - 2q_4^2)}} \\ &= -\hat{\boldsymbol{v}} \pm \frac{-q_4\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}} + 2\cos^2(\theta/2)\boldsymbol{q}_{1:3}}{\sqrt{\cos^2(\theta/2) - q_4^2}} \end{aligned} \tag{3}$$

This can be solved for $\boldsymbol{q}_{1:3}$ in terms of $\hat{\boldsymbol{v}}, \hat{\boldsymbol{u}}$, and q_4. Substituting $q_4 = s$ the quaternions satisfying Eq. (2) are

$$\begin{aligned} \boldsymbol{f}^s(s) &= \begin{bmatrix} \frac{s(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}}) \pm \hat{\boldsymbol{v}} + (\hat{\boldsymbol{u}}/\sqrt{2})(1 + \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}} - 2s^2)^{1/2}}{1 + \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}}} \\ s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}}) \pm (\hat{\boldsymbol{v}} + \hat{\boldsymbol{u}})\cos(\theta/2)\sqrt{1 - \left(s/\cos(\theta/2)\right)^2}}{2\cos(\theta/2)} \\ s \end{bmatrix} \end{aligned} \tag{4}$$

Inserting Eq. (4) into Eq. (2) shows that it is correct. Substitutions lead to advantageous formulations, shown in Table 1. For example, the structure of Eq. (4) allows the simultaneous substitution of

$$\cos\beta = \frac{s}{\cos(\theta/2)} \quad \text{and} \quad \sin\beta = \pm\sqrt{1 - \left(\frac{s}{\cos(\theta/2)}\right)^2}$$

with "$\pm$" consumed by the $\sin\beta$ sign change (with $\cos(\theta/2) \ge 0$ by the definition of θ). This yields

[‡]This Note mostly avoids the use of inverse trigonometric functions, due to their restricted codomain and the necessity of allowing for sign changes or shifts. However, in this case, the function limits map exactly onto the space of possibilities.

Table 1 Formulations of quaternion set

Variable	Definition	Reference	Bounds	Advantage
s	$\begin{bmatrix} \frac{s(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})\pm((\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})/\sqrt{2})(1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}-2\,s^2)^{1/2}}{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}} \\ s \end{bmatrix}$	Eq. (4)	$-\sqrt{\frac{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}}{2}} \le s \le \sqrt{\frac{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}}{2}}$	$s = q_4$
β	$\boldsymbol{y}\sin\beta + \boldsymbol{x}\cos\beta$	Eq. (5)	$\beta \in \mathbb{R}$	Continuous, simple
ϕ	$\begin{bmatrix} \frac{\cos(\phi/2)(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})\pm(\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})\left(\cos^2(\phi/2)-\cos^2(\phi/2)\right)^{1/2}}{2\cos^2(\theta/2)} \\ \cos\frac{\phi}{2} \end{bmatrix}$	Eq. (7)	$\arccos(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}) \le \lvert\phi\rvert \le \pi$	ϕ is rotation angle
α	$\frac{\boldsymbol{x}\cos\alpha+\boldsymbol{y}\sin(\theta/2)\sin\alpha}{\sqrt{\cos^2\alpha\cos^2(\theta/2)+\sin^2(\theta/2)}}$	Eq. (9)	$\alpha \in \mathbb{R}$	α indexes rotation axis

$$\boldsymbol{f}^{\beta}(\beta) = \boldsymbol{y}\sin\beta + \boldsymbol{x}\cos\beta \tag{5}$$

where

$$\boldsymbol{x} = \left[\frac{(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})^T}{2\cos(\theta/2)}\cos\frac{\theta}{2}\right]^T \text{ and } \boldsymbol{y} = \left[\frac{(\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})^T}{2\cos(\theta/2)}0\right]^T \tag{6}$$

Applying Eq. (1) shows the significance: $\boldsymbol{y}$ corresponds to a rotation of π around $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})/(\|\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}}\|)$ and $\boldsymbol{x}$ is the minimal rotation of θ around $\hat{\boldsymbol{m}} = (\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})/(\|\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}}\|)$. This formulation is related to the set of Eq. (16) in [13], with conversion $\varphi = 2\beta$.

Additionally, there can be physical significance to the indexing. We can have the index variable be the rotation angle of the quaternion's axis-angle form [3]. We apply the variable substitution $s = \cos(\phi/2)$ to Eq. (4) to find

$$\begin{aligned}\boldsymbol{f}^{\phi}(\phi) &= \begin{bmatrix} \frac{\cos(\phi/2)(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})\pm((\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})/\sqrt{2})\left(1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}-2\cos^2(\phi/2)\right)^{1/2}}{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}} \\ \cos\frac{\phi}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\cos(\phi/2)(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})\pm(\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})\left(\cos^2(\theta/2)-\cos^2(\phi/2)\right)^{1/2}}{2\cos^2(\theta/2)} \\ \cos\frac{\phi}{2} \end{bmatrix}\end{aligned} \tag{7}$$

Substituting this into Eq. (1), we see that it indexes the set of acceptable orientations with the single-axis rotation angle.

Lastly, like Eq. (7) indexes the rotation angle, there is a formulation where the quaternion set is indexed based on the rotation axis. Express that axis on the unit circle defined by $\hat{\boldsymbol{m}} = (\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}})/(\|\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}}\|)$ and $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}})/(\|\hat{\boldsymbol{v}}+\hat{\boldsymbol{u}}\|)$, indexed by α. By Eq. (1),

$$\boldsymbol{f}^{\alpha}(\alpha) = \left[\sin\frac{\gamma(\alpha)}{2}(\hat{\boldsymbol{m}}\cos\alpha + \hat{\boldsymbol{n}}\sin\alpha)^T\cos\frac{\gamma(\alpha)}{2}\right]^T$$

where the rotation angle $\gamma(\alpha)$ is a function of α. Equating this with Eq. (5) or Eq. (7) and solving gives

$$\tan\frac{\gamma(\alpha)}{2} = \frac{\tan(\theta/2)}{\cos\alpha} \tag{8}$$

To make $\gamma(\alpha) \ge 0$, we solve this as is:

$$\gamma(\alpha) = \pi - 2\arctan\left(\cos\alpha\cot\frac{\theta}{2}\right)$$

Our axis-based formulation is

$$\begin{aligned}\boldsymbol{f}^{\alpha}(\alpha) &= \frac{1}{\sqrt{\cos^2\alpha+\tan^2(\theta/2)}}\begin{bmatrix}\tan\frac{\theta}{2}(\hat{\boldsymbol{m}}\cos\alpha+\hat{\boldsymbol{n}}\sin\alpha)\\ \cos\alpha\end{bmatrix} \\ &= \frac{\boldsymbol{x}\cos\alpha+\boldsymbol{y}\sin(\theta/2)\sin\alpha}{\sqrt{\cos^2\alpha\cos^2(\theta/2)+\sin^2(\theta/2)}}\end{aligned} \tag{9}$$

III. Minimum Slew from Arbitrary Quaternion

The formulas from Sec. II describe the set of quaternions that align the vectors. Here, we will identify which quaternions in the set are closest to a given quaternion. This requires a distance metric. Consider quaternions $\boldsymbol{p}$ and $\boldsymbol{q}$ separated by angle δ, the rotation angle of $\boldsymbol{p}\boldsymbol{q}^*$. From quaternion multiplication and Eq. (1) (and borrowing the vector dot product), $\cos(\delta/2) = (\boldsymbol{p}\boldsymbol{q}^*)_4 = \boldsymbol{p}_{1:3}\cdot\boldsymbol{q}_{1:3} + p_4 q_4 = \boldsymbol{p}\cdot\boldsymbol{q}$. The cosine double-angle identity then yields

$$\text{Distance Metric: } \Delta(\boldsymbol{p},\boldsymbol{q}) = \arccos(2(\boldsymbol{p}\cdot\boldsymbol{q})^2 - 1) \tag{10}$$

Consider the input quaternion $\boldsymbol{p} = [\,\boldsymbol{p}_{1:3}^T \quad p_4\,]^T$. The quaternion that meets the pointing goals that is nearest [by Eq. (10)] $\boldsymbol{p}$ is an orientation a satellite can assume to meet the goal with minimal slew. Using Eq. (10), the distance between $\boldsymbol{p}$ and the quaternions in Eq. (5) is $\Delta(\boldsymbol{p}, \boldsymbol{f}^{\beta}(\beta))$, so minima will satisfy

$$0 = \frac{d\Delta}{d\beta} = -\frac{4(\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta))\left(\boldsymbol{p}\cdot\frac{d\boldsymbol{f}^{\beta}(\beta)}{d\beta}\right)}{\sqrt{1-(2(\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta))^2-1)^2}} \tag{11}$$

This is necessary but not sufficient, so the solutions must be checked. One solution is $\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta) = 0$, which, with Eq. (10), yields a maximized distance of π between $\boldsymbol{f}^{\beta}(\beta)$ and $\boldsymbol{p}$. Solving $\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta) = 0$ and Eq. (5) gives

$$\tan\beta_0 = \frac{-\boldsymbol{p}\cdot\boldsymbol{x}}{\boldsymbol{p}\cdot\boldsymbol{y}} \text{ and } \boldsymbol{f}^{\beta}(\beta_0) = \pm\frac{\boldsymbol{x}(\boldsymbol{p}\cdot\boldsymbol{y}) - \boldsymbol{y}(\boldsymbol{p}\cdot\boldsymbol{x})}{\sqrt{(\boldsymbol{p}\cdot\boldsymbol{x})^2+(\boldsymbol{p}\cdot\boldsymbol{y})^2}} \tag{12}$$

The other simple case is when $\boldsymbol{p}\cdot(d\boldsymbol{f}^{\beta}(\beta)/d\beta) = 0$. Differentiating Eq. (5) gives $(d\boldsymbol{f}^{\beta}(\beta)/d\beta) = \boldsymbol{y}\cos\beta - \boldsymbol{x}\sin\beta$, so this case occurs if $\boldsymbol{p}\cdot\boldsymbol{y}\cos\beta = \boldsymbol{p}\cdot\boldsymbol{x}\sin\beta$. Solving this with Eq. (5) yields the nearest quaternion to $\boldsymbol{p}$:

$$\tan\beta_1 = \frac{\boldsymbol{p}\cdot\boldsymbol{y}}{\boldsymbol{p}\cdot\boldsymbol{x}} \text{ and } \boldsymbol{f}^{\beta}(\beta_1) = \pm\frac{(\boldsymbol{p}\cdot\boldsymbol{x})\boldsymbol{x} + (\boldsymbol{p}\cdot\boldsymbol{y})\boldsymbol{y}}{\sqrt{(\boldsymbol{p}\cdot\boldsymbol{x})^2+(\boldsymbol{p}\cdot\boldsymbol{y})^2}} \tag{13}$$

The distance between this nearest quaternion and $\boldsymbol{p}$ is $\Delta = \arccos(-1 + 2((\boldsymbol{p}\cdot\boldsymbol{x})^2 + (\boldsymbol{p}\cdot\boldsymbol{y})^2))$, the minimal distance.

The values from Eqs. (12) and (13) work for almost any $\boldsymbol{p}$. A few exceptions should be checked. If $\boldsymbol{p}\cdot\boldsymbol{x} = \boldsymbol{p}\cdot\boldsymbol{y} = 0$, all quaternions in the set $\boldsymbol{f}^{\beta}(\beta)$ are a distance of π from $\boldsymbol{p}$. The denominator singularity can be safely ignored, as it simplifies to $2(\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta))^2 - 1 = \pm 1$, leaving two cases: $\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta) = 0$ and $\boldsymbol{p}\cdot\boldsymbol{f}^{\beta}(\beta) = \pm 1$. The former was already explored and leads to Eq. (12). The latter has $\Delta = 0$ distance but only happens if the set $\boldsymbol{f}^{\beta}(\beta)$ already contains $\boldsymbol{p}$. It is also worth noting that if $\boldsymbol{p}$ is the identity quaternion, then $\boldsymbol{x}$ is the nearest quaternion to $\boldsymbol{p}$ in the set and $\boldsymbol{y}$ is the farthest.

IV. Slewing with "Keep-Out" Zone

The use of these quaternion sets also increases flexibility and allows for more complex goals, such as single-axis slews that still respect a keep-out zone. In addition to $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{u}}$, consider $\hat{\boldsymbol{b}}$, a body

Downloaded by 2601:184:4980:fb90:dd06:a50f:2d31:1e1a on May 21, 2026 | http://arc.aiaa.org | DOI: 10.2514/1.G006429

Downloaded by 2601:184:4980:fb90:dd06:a50f:2d31:1e1a on May 21, 2026 | http://arc.aiaa.org | DOI: 10.2514/1.G006429

frame vector that we want to "keep out" of a zone around a certain world-frame direction $\hat{\boldsymbol{a}}$. The simplest example would be an imager that should not point at the sun. Specifically, we do not want $\hat{\boldsymbol{b}}$ to be within angle η of $\hat{\boldsymbol{a}}$. This constraint can be tested on an orientation $\boldsymbol{q}$ using Eq. (2) to give $\hat{\boldsymbol{a}}^T A(\boldsymbol{q}^*)\hat{\boldsymbol{b}} \leq \cos\eta$. One approach to go between a starting orientation $\boldsymbol{p}$ and some specific ending orientation, while avoiding the keep-out zone, would be to use sequential nonaligned slews. However, the use of a set of goals allows this to be considered in terms of single-axis rotations.

We want to find which single-axis slews between starting orientation $\boldsymbol{p}$ and goal set $\boldsymbol{f}(\alpha)$ [using Eq. (9) as the rotation axis $\hat{\boldsymbol{e}}(\alpha) = \hat{\boldsymbol{m}}\cos\alpha + \hat{\boldsymbol{n}}\sin\alpha$ is relevant] avoid the keep-out zone. Without loss of generality, assume a starting orientation of $\boldsymbol{p} = [\,\boldsymbol{0}^T \quad 1\,]^T$. (For an arbitrary starting orientation $\boldsymbol{p}_0$, first rotate $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{a}}$ by $\boldsymbol{p}_0^*$.) The slew to $\boldsymbol{f}(\alpha)$ then simplifies to $\boldsymbol{p}^*\boldsymbol{f}(\alpha) = \boldsymbol{f}(\alpha)$ (as $\boldsymbol{p}$ is the identity quaternion). For later concision, define the values $i(\alpha) = (\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{e}}(\alpha))(\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{e}}(\alpha))$, $j(\alpha) = (\hat{\boldsymbol{a}}\times\hat{\boldsymbol{e}}(\alpha))\cdot(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{e}}(\alpha))$, and $k(\alpha) = \hat{\boldsymbol{e}}(\alpha)\cdot(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})$.

First, we need to know that the keep-out zone is not violated at the starting or ending orientation, so we need both $\hat{\boldsymbol{a}}^T A(\boldsymbol{p}^*)\hat{\boldsymbol{b}} \leq \cos\eta$ and $\hat{\boldsymbol{a}}^T A(\boldsymbol{f}(\alpha)^*)\hat{\boldsymbol{b}} \leq \cos\eta$. Applying Eq. (2) to these yields the conditions (leaving off the α argument from i, j, and k)

$$i + j = \hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}} \leq \cos\eta \tag{14a}$$

$$i + j - \frac{2\tan(\theta/2)\Big(k\cos\alpha + j\tan(\theta/2)\Big)}{\cos^2\alpha + \tan^2(\theta/2)} \leq \cos\eta \tag{14b}$$

However, it is also possible that the starting and ending orientations avoid the keep-out zone but it is violated midslew. During the slew around the axis $\hat{\boldsymbol{e}}(\alpha)$, the rotation angle ρ will go from 0 to $\gamma(\alpha)$ [Eq. (8)]. From Eq. (1), the orientation midslew is

$$\boldsymbol{q}(\rho) = \begin{bmatrix} \sin\frac{\rho}{2}\hat{\boldsymbol{e}}^T(\alpha) & \cos\frac{\rho}{2} \end{bmatrix}^T$$

We need $\hat{\boldsymbol{a}}^T A(\boldsymbol{q}^*(\rho))\hat{\boldsymbol{b}} \leq \cos\eta$ for $0 < \rho < \gamma(\alpha)$. From Eq. (2) (or the Rodrigues rotation formula), this condition expands to $\cos\eta \geq i(\alpha) + j(\alpha)\cos\rho - k(\alpha)\sin\rho$. For this condition to pass, there are two (nonexclusive) possibilities: either the global maximum of the expression is less than $\cos\eta$, or the maximum occurs outside of $0 < \rho < \gamma(\alpha)$. As the expression has the form $a + b\cos x + c\sin x$ (which has its maximum at $x = \text{atan2}(c, b)$), we know that our maxima are at

$$\rho_{\max} = \text{atan2}(-k, j) = 2\arctan\left(\frac{-k}{j + \sqrt{k^2 + j^2}}\right)$$

To ensure $\rho_{\max} \geq 0$, we rewrite this as

$$\rho_{\max} = \pi + 2\arctan\left(\frac{j + \sqrt{k^2 + j^2}}{k}\right)$$

The value at this maximum is $\hat{\boldsymbol{a}}^T A(\boldsymbol{q}^*(\rho_{\max}))\hat{\boldsymbol{b}} = i + \sqrt{j^2 + k^2}$. Thus, for an acceptable slew, one or both of the conditions must be met: the maximum does not violate the constraint $\cos\eta \geq \hat{\boldsymbol{a}}^T A(\boldsymbol{q}^*(\rho_{\max}))\hat{\boldsymbol{b}}$, or this maximum occurs beyond the bounds of the slew angle:

$$\rho_{\max} > \gamma(\alpha) = \pi - 2\arctan\left(\cos\alpha\cot\frac{\theta}{2}\right)$$

Rewriting these conditions yields

$$\cos\eta \geq i + \sqrt{j^2 + k^2} \tag{15a}$$

$$\frac{j + \sqrt{j^2 + k^2}}{k} > -\cos\alpha\cot\frac{\theta}{2} \tag{15b}$$

Thus, values of α satisfying the conditions in Eqs. (14) and (15) correspond to acceptable slews. This is a constrained optimization problem, and the equations involved are best solved graphically or numerically. One option is provided in this Note's associated Github Repo. It is also worth noting that there may not always be a solution. This can occur, for example, if the keep-out zone is violated at $\rho = 0$, making Eq. (14a) always fail. Similarly, if Eq. (14b) has no feasible solution, all options in the goal set violate the constraint and there is no solution.

V. Application Cases

The simplest example case is to point a spacecraft with minimum slew. Consider a directed antenna boresight along the body x axis that needs to point at a ground station in the world-frame direction $(1/\sqrt{2})[\,0 \quad 1 \quad -1\,]^T$. This yields $\hat{\boldsymbol{v}} = [\,1 \quad 0 \quad 0\,]^T$ and $\hat{\boldsymbol{u}} = (1/\sqrt{2})[\,0 \quad 1 \quad -1\,]^T$, so $\theta = (\pi/2)$. Additionally,

$$\boldsymbol{x} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \quad \text{and} \quad \boldsymbol{y} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1 & -1 & 0 \end{bmatrix}^T$$

The set of orientations that point $\hat{\boldsymbol{v}}$ along $\hat{\boldsymbol{u}}$ are thus, from Eq. (5),

$$\boldsymbol{f}^\beta(\beta) = \frac{1}{2}\begin{bmatrix} \sqrt{2}\sin\beta\sin\beta + \cos\beta\cos\beta - \sin\beta\sqrt{2}\cos\beta \end{bmatrix}^T$$

First, a simple case, with the satellite's current orientation $\boldsymbol{p} = [\,\boldsymbol{0}^T \quad 1\,]^T$. Using Eq. (13), we find that the acceptable orientation closest to this orientation is $\boldsymbol{x}$, with $\beta = 0$. The rotation required is $\boldsymbol{p}^*\boldsymbol{x} = \boldsymbol{x}$, which has an angle of $(\pi/2)$ about the axis $(1/\sqrt{2})[\,0 \quad 1 \quad 1\,]^T$ in both the body and world frames.

Now consider $\boldsymbol{p} = (1/5)[\,1 \quad -2 \quad 2 \quad 4\,]^T$. From Eq. (13), the acceptable orientation closest to $\boldsymbol{p}$ is

$$\frac{4\boldsymbol{x} + (1 - 2\sqrt{2})\boldsymbol{y}}{\sqrt{25 - 4\sqrt{2}}}$$

with $\beta \approxeq -1.142$. The rotation required has an angle of ≈ 1.799 rad about the body-frame axis $(1/\sqrt{1186})[\,0 \quad 31 \quad 15\,]^T$.

The ability to use a variety of goal orientations can be useful for other trajectory goals. Consider the trajectory problem, with a $\eta = 30°$ keep-out zone to avoid pointing an imager toward the sun. The imager boresight is $\hat{\boldsymbol{b}} = [\,0 \quad 1 \quad 0\,]^T$ and the sun is in the direction $\hat{\boldsymbol{a}} = (1/2)[\,-\sqrt{3} \quad 1 \quad 0\,]^T$ in the world frame. We will use $\boldsymbol{p}_0 = [\,\boldsymbol{0}^T \quad 1\,]^T$. The axis of rotation will be $\hat{\boldsymbol{e}}(\alpha) = \hat{\boldsymbol{m}}\cos\alpha + \hat{\boldsymbol{n}}\sin\alpha$, where

$$\hat{\boldsymbol{m}} = \frac{\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}}}{\|\hat{\boldsymbol{v}}\times\hat{\boldsymbol{u}}\|} = \frac{1}{\sqrt{2}}[\,0 \quad 1 \quad 1\,]^T \quad \text{and} \quad \hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{v}} + \hat{\boldsymbol{u}}}{\|\hat{\boldsymbol{v}} + \hat{\boldsymbol{u}}\|} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1 & -1 \end{bmatrix}^T$$

The angle of rotation is $\gamma(\alpha) = \pi - 2\arctan(\cos\alpha)$. The trajectory must satisfy Eqs. (14) and (15), which are best solved graphically or numerically. Figure 1 shows the plot of the relevant inequality bounds. The conditions in Eqs. (14a) and (14b) are solid lines, which both must be negative. Those in Eqs. (15a) and (15b) are dash-dot lines; at least one must be negative. With these constraints, Fig. 1 shows that acceptable α are $0.094 \lessapprox \alpha \lessapprox 2.610$ and $3.138 \lessapprox \alpha \lessapprox 5.327$. Note that the minimum slew ($\alpha = 0$) is not acceptable, because both $\rho_{\max} < \gamma$ [Eq. (15b)] and $\hat{\boldsymbol{a}}^T A(\boldsymbol{q}^*(\rho_{\max}))\hat{\boldsymbol{b}} \geq \cos\eta$ [Eq. (15a)], so the keep-out zone is violated. Instead, $\alpha \approxeq 0.094$ has the minimum slew of ≈ 1.575 rad (only slightly more than the minimum possible slew of $\pi/2$).

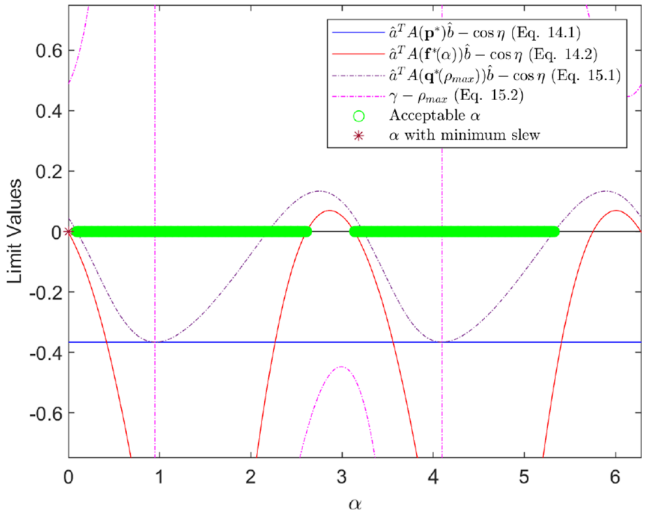


Fig. 1 Plot of constraints that must be satisfied to avoid keep-out zone. Lines represent inequality bounds: both solid lines and one or more dash-dot line must be negative for allowed rotations.

VI. Conclusions

Specifying an exact orientation may be an unnecessary constraint for pointing spacecraft if the goal is based on a single axis. The various formulas in Sec. II describe a set of orientations that could point the spacecraft correctly. In many cases, the most useful member of this set will be the one closest to a satellite's current orientation—the formula for the orientation with minimum slew from an arbitrary quaternion is found in Sec. III. However, the extra degree of freedom provided by using a set of orientations (instead of simply the closest one) allows for consideration of additional criteria. Section IV shows an example of this, by finding the slews that avoid a keep-out zone. Finally, examples of these formulas in use were shown in the application cases in Sec. V.

Acknowledgments

This work was completed while supported by MIT Portugal: Portuguese Science and Technology Foundation for work on the project "AEROS Constellation—Development of a Nanosatellite Platform as a Precursor of a Future Constellation to Leverage the Space/Ocean Scientific and Economic Synergies"; MIT Lincoln Laboratory for work on the project "Time-Resolved Observations of Precipitation Structure and Storm Intensity with a Constellation of Smallsats (TROPICS)"; and Northrop Grumman for work on the project "MIT BeaverCube II."

References

[1] Bilimoria, K. D., and Wie, B., "Time-Optimal Three-Axis Reorientation of a Rigid Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 3, 1993, pp. 446–452. https://doi.org/10.2514/3.21030

[2] Yang, Y., "Spacecraft Attitude Determination and Control: Quaternion Based Method," *Annual Reviews in Control*, Vol. 36, No. 2, 2012, pp. 198–219. https://doi.org/10.1016/j.arcontrol.2012.09.003

[3] Markley, F. L., and Crassidis, J. L., *Fundamentals of Spacecraft Attitude Determination and Control*, Springer, New York, 2014, Chap. 2. https://doi.org/10.1007/978-1-4939-0802-8

[4] Wie, B., and Lu, J., "Feedback Control Logic for Spacecraft Eigenaxis Rotations Under Slew Rate and Control Constraints," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 6, 1995, pp. 1372–1379. https://doi.org/10.2514/3.21555

[5] Crassidis, J. L., and Markley, F. L., "Sliding Mode Control Using Modified Rodrigues Parameters," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 6, 1996, pp. 1381–1383. https://doi.org/10.2514/3.21798

[6] Bullo, F., Murray, R. M., and Sarti, A., "Control on the Sphere and Reduced Attitude Stabilization," *IFAC Proceedings Volumes*, Vol. 28, No. 14, 1995, pp. 495–501. https://doi.org/10.1016/S1474-6670(17)46878-9

[7] Chaturvedi, N. A., Sanyal, A. K., and McClamroch, N. H., "Rigid-Body Attitude Control," *IEEE Control Systems Magazine*, Vol. 31, No. 3, 2011, pp. 30–51. https://doi.org/10.1109/MCS.2011.940459

[8] de Angelis, E. L., Giulietti, F., and Avanzini, G., "Single-Axis Pointing of Underactuated Spacecraft in the Presence of Path Constraints," *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 1, 2015, pp. 143–147. https://doi.org/10.2514/1.G000121

[9] Pong, C. M., and Miller, D. W., "Reduced-Attitude Boresight Guidance and Control on Spacecraft for Pointing, Tracking, and Searching," *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 6, 2015, pp. 1027–1035. https://doi.org/10.2514/1.G000264

[10] Markdahl, J., Hoppe, J., Wang, L., and Hu, X., "A Geodesic Feedback Law to Decouple the Full and Reduced Attitude," *Systems & Control Letters*, Vol. 102, April 2017, pp. 32–41. https://doi.org/10.1016/j.sysconle.2017.01.005

[11] Hu, Q., Chi, B., and Akella, M. R., "Reduced Attitude Control for Boresight Alignment with Dynamic Pointing Constraints," *IEEE/ASME Transactions on Mechatronics*, Vol. 24, No. 6, 2019, pp. 2942–2952. https://doi.org/10.1109/TMECH.2019.2944431

[12] Dai, J. S., "Euler–Rodrigues Formula Variations, Quaternion Conjugation and Intrinsic Connections," *Mechanism and Machine Theory*, Vol. 92, Oct. 2015, pp. 144–152. https://doi.org/10.1016/j.mechmachtheory.2015.03.004

[13] Markley, L., "Fast Quaternion Attitude Estimation from Two Vector Measurements," *Journal of Guidance, Control, and Dynamics*, Vol. 25, March 2002, pp. 411–414. https://doi.org/10.2514/2.4897

[14] Huynh, D. Q., "Metrics for 3D Rotations: Comparison and Analysis," *Journal of Mathematical Imaging and Vision*, Vol. 35, No. 2, 2009, pp. 155–164. https://doi.org/10.1007/s10851-009-0161-2

Downloaded by 2601:184:4980:fb90:dd06:a50f:2d31:1e1a on May 21, 2026 | http://arc.aiaa.org | DOI: 10.2514/1.G006429